Analyzing higher order gravitational interactions

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Abstract

Perturbation theory allows the equations of motion for planets to be realized, making use of derived first-order perturbation equations and ultimately extending them to higher orders of perturbations. This allows for accurate predictions of the positions of celestial bodies without needing to resort to more difficult methods of solving complex systems. By assuming circular orbits the problem becomes simplified and solvable, which can consequently be tweaked in order to achieve more accurate results.

1 Introduction

Perturbation theory is at the forefront of applied physics and mathematics, as it allows for approximate solutions to be found for problems that may not be analytically solvable. By slightly tweaking the parameters of a simplified problem, an iteratively more accurate solution can be found. A prime example of the use of perturbation theory is the theoretical prediction of the location of Neptune, which was later confirmed. This was done by the mathematician Urbain Le Verrier, who made significant use of perturbation theory in his calculations [1]. This report will analyze a simplified model of celestial bodies and solve a problem similar to the one solved by Le Verrier using the theory of perturbation.

2 Theory

To describe the motion of an planet, a complex system of differential equations has to be solved. To solve these problems the perturbation theory is used in order to simplify and approximate the solutions of the system. In practice, start with an simplified version of the problem and make corrections to make it closer to reality.

2.1 The dynamics

Consider two planets A,B with masses M_A,M_B that orbit the Sun. Assume the Sun as unmovable and the planets move in the same plane, so their coordinates

are $(x_A, x_B), (y_A, y_B)$. If we consider the origin at the Sun, we can describe the motion of planet A as:

$$\ddot{x}_A = -gM_S \frac{x_A}{R_A^3} - gM_B \frac{x_A - x_B}{R_{AB}^3} \tag{1}$$

$$\ddot{y}_A = -gM_S \frac{y_A}{R_A^3} - gM_B \frac{y_A - y_B}{R_{AB}^3} \tag{2}$$

With g the gravitational constant, M_S the mass of the Sun, $R_A = \sqrt{x_A^2 + y_A^2}$ the distance from the Sun to planet A and $R_{AB} = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$ the distance between the planets A,B. The same applies for planet B [2].

2.2 The zero-order solution

The first simplification comes from the fact that the mass from the Sun is much greater than the masses of the planets A and B, $M_S >> M_A, M_B$. In this case the mutual forces can be neglected against the force of the Sun. The solution is called the zero-order solution and it describes the planets in a circular motion. Writing the equations in polar coordinates simplifies this problem. The following solutions are presented in the zero-order solution:

$$x_A, y_A = (sin(\phi_A^{(0)}), cos(\phi_A^{(0)}))R_A^{(0)} \tag{3}$$

$$x_B, y_B = (sin(\phi_B^{(0)}), cos(\phi_B^{(0)}))R_B^{(0)} \tag{4}$$

$R_A^{(0)}$ does not depend on time while:

$$\phi_A^{(0)}(t) = \Omega_A t; \phi_B^{(0)}(t) = \Omega_B t + \psi \tag{5}$$

3 Method

To enhance the accuracy of the zero-order solution, it is essential to incorporate corrections that refine our approximation. These corrections allow for a more realistic representation of the system's dynamics. The first-order correction will be described in the following five steps.

The first step is to rewrite the equations from Cartesian coordinates to polar coordinates. This transformation is described as follows: $(x_A, y_A) \rightarrow (R_A \sin(\phi_A), R_A \cos(\phi_A))$ the same can be done for planet B.

The second step the mutual forces will be includes in the solution. It is crucial to emphasize that this step will build upon the assumptions established in the zero-order solution. Since the dynamics are now in Polar coordinates, the force is computed in radial and tangential components, $F^{r,t}(t)$. With the use of the radial and tangential acceleration, two differential equations are denoted as functions of time.

$$F_A^r(t) = \ddot{R}_A - \dot{\phi}^2 R_A \tag{6}$$

$$F_A^t(t) = R_A\ddot{\phi} + 2\dot{\phi}R_A \tag{7}$$

In the next step, the equations are expended with the deviations of the zero-order approximation: $\delta r_{A,B}(t) = R_{A,B}(t) - R^{(0)}; \delta\phi_{A,B}(t) = \phi_{A,B}(t) - \phi_{A,B}^{(0)}$. After linearization two inhomogeneous differential equations are found for $\delta r_{A,B}(t), \delta\phi_{A,B}(t)$.

The fourth step is the analyzation of the time-dependence of F_A, were the conclusion can be made to present the forces in an Fourier series:

$$F_r^A(t) = \sum_{n=0}^{\infty} F_{r;A;n} \cos\left(n(\Omega_A - \Omega_B)t - \psi\right) \tag{8}$$

$$F_t^A(t) = \sum_{n=0}^{\infty} F_{t;A;n} \cos\left(n(\Omega_A - \Omega_B)t - \psi\right) \tag{9}$$

With the Fourier coefficients:

$$F_{r,t;A;n} = \frac{1}{T}\int_{\langle T\rangle} F_{r,t}^A cos(n(\Omega_A - \Omega_B)t + \psi) \tag{10}$$

Were $T = \frac{2\pi}{\Omega}$ and n is the number of harmonics. With this we can solve for the differential equations (6) and (7).

The same is possible for the equations of $\delta r_{A,B}(t)$ and $\delta\phi_{A,B}(t)$. With this, the differential equations can be solved.

$$\delta r_{A,B}(t) = \sum_{n=0}^{\infty} c_{r;n} \cos\left(n(\Omega_A - \Omega_B)t - \psi\right) \tag{11}$$

$$\delta\phi_{A,B} = \sum_{n=0}^{\infty} c_{\phi;n} \cos\left(n(\Omega_A - \Omega_B)t - \psi\right) \tag{12}$$

The Fourier coefficients $c_{\phi;n}$ and $c_{r;n}$ can be expressed in terms of $F_{a;n}^r$ and $F_{a;n}^t$.

4 Results

4.1 Results First order perturbation

By rewriting the forces in polar coordinates and filling it in equations 6 and 7. As explained in the method, the equations can be expanded with the deviations of the zero-order approximation. The following two inhomogeneous differential equations are found:

$$F_A^r(t) = \delta\ddot{r}_A - \Omega_A^2\,\delta r_A - 2\Omega_A R_A^{(0)}\,\delta\dot{\phi}_A = -gM_B\frac{R_A^{(0)} - R_B^{(0)}cos(\Omega t - \psi)}{R_{AB}^3} \tag{13}$$

$$F_A^t(t) = R_A^{(0)}\,\delta\ddot{\phi}_A + 2\Omega_A\,\delta\dot{r}_A = -gM_B\frac{R_B^{(0)}sin(\Omega t - \psi)}{R_{AB}^3} \tag{14}$$

Where $R_{AB} = \sqrt{R_A^2 + R_B^2 - 2R_AR_Bcos(\Omega t - \psi)}$ and $\Omega = \Omega_A - \Omega_B$. Now the equations 8,9,11 and 12 can be used to solve for the Fourier coefficients $c_{\phi;n}$ and $c_{r;n}$.

$$c_{r;n} = \frac{-n\Omega F_{a;n}^r + 2\Omega_A F_{A;n}^t}{n\Omega(n^2\Omega^2 - \Omega_A^2)} \tag{15}$$

$$c_{\phi;n} = -\frac{F_{t;n}^t(n^2\Omega^2 + \Omega_A^2) + 4\Omega\Omega_A F_{A;n}^r}{n^2\Omega^2 R_A^{(0)}(n^2\Omega^2 - \Omega_A^2)} \tag{16}$$

4.2 Determining orbit planet B

Determining the orbit of planet B is also possible. For this we will use a simplified model in which the tangential force can be neglected, or in other words, the tangential force F^t is equal to zero. This is possible when the planets and the sun are aligned on one axis. Furthermore, if the two planets are on the same of the sun the proximity between the planets R_{AB} are at a minimum. What follows is that the radial component of the mutual force is therefore maximal. Recalling Kepler's third law, the following relation can be found between the period T and the distance R:

$$T = \sqrt{\frac{4\pi^2}{GM}(R_B^{(0)})^3} \tag{17}$$

Using this value for T in the equation for the angular frequency $\Omega = \frac{2\pi}{T}$ yields the following equation for $R_B^{(0)}$, which can ultimately be used to explain the movement of planet B.

$$R_B^{(0)} = \sqrt[3]{\frac{GM_s}{\Omega - \Omega_A}} \tag{18}$$

4.3 Higher order perturbations

For higher order perturbations, the coordinates of the respective planets x can be expressed as a series expansion:

$$x(t) = x^{(0)}(t) + \epsilon x^{(1)}(t) + \epsilon^2 x^{(2)}(t) + ... \tag{19}$$

$$y(t) = y^{(0)}(t) + \epsilon y^{(1)}(t) + \epsilon^2 y^{(2)}(t) + ... \tag{20}$$

Where ϵ is a constant proportional to $\frac{M}{M_S}$ with M the mass of the planet in question and $x^{(n)}, y^{(n)}$ the n-th order perturbative corrections.

These can be substituted in the equation of motion (1) and (2), which results in new equations which can be identified and solved to achieve the n-th order corrections. After summing the results, a higher order solution can be found. Iteratively this can be repeated by substituting $x^{(n)}, y^{(n)}$ into $x^{(n+1)}, y^{(n+1)}$.

5 Bibliography

[1] Adams, J.C., MA, FRAS, Fellow of St John's College, Cambridge (1846). "On the Perturbations of Uranus". UK Nautical Almanac Office, 1851. p. 265.

[2] Yuli Nazarov. With the point of your pen. TU Delft, 2024.